

Nonlinear Debye screening in strongly-coupled plasmas

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Abstract

An ubiquitous property of plasmas is the so-called Debye shielding of the electrostatic potential. Important aspects of Debye screening concern, in particular, the investigation of non-linear charge screening effects taking place in strongly-coupled plasmas, that imply a reduction of the effective charge characterizing the Debye-Hückel potential. These effects are particularly relevant in dusty plasmas which are characterized by high-Z particles. The investigation of the effective interactions of these particles has attracted interest in recent years especially for numerical simulations. In this work we intend to analyze the consistency of the traditional mathematical model for the Debye screening. In particular, we intend to prove that the 3D Poisson equation involved in the DH model does not admit strong solutions. For this purpose a modified model is proposed which takes into account the effect of local plasma sheath (i.e., the local domain near test particles where the plasma must be considered discrete). Basic consequences of the model are discussed, which concern the asymptotic properties of the solutions determined both for weakly and strongly coupled plasmas. As an application the charge screening effect in strongly coupled plasmas is investigated and an explicit expression of the effective charge for the asymptotic DH potential is determined.

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I. INTRODUCTION

A basic aspect of plasma physics is the so-called Debye shielding of the electrostatic potential. This consists in the property of plasmas (or electrolytes [1]), either quasi-neutral or non-neutral, to shield the electrostatic field produced by charged particles (to be also denoted as test particles) immersed in the same system. This result has fundamental consequences on plasma phenomenology, since it actually limits the range of static Coulomb interactions inside the Debye sphere, i.e., at a distance $\rho \leq \lambda_D$ from the test particle, λ_D being the Debye length. As usual here $\lambda_D \equiv \left(\sum_s \lambda_{Ds}^{-1} \right)^{-1}$, where the sum is carried out on all plasma species and $\lambda_{Ds} = \sqrt{\frac{T_s}{4\pi Z_s^2 e^2 N_{os}}}$, T_s and N_{os} being respectively the s -species temperature and number density (the latter defined in the absence of test particles). In fact, when both the particles and the plasma are assumed non-relativistic, small-amplitude, stationary (or slowly time- and space-varying), electrostatic perturbations generated by isolated test particles, result effectively shielded in the external domain, i.e., at distances larger than the Debye length λ_D . The renewed interest in this problem [2, 3, 4, 5, 6, 7, 8, 9] is particularly related to dusty plasmas or colloidal suspensions which are characterized by the presence of a large fraction of highly charged particles (grains), i.e., having an electric charge $Z_d e$ with $|Z_d| \gg 1$.

In this work we intend to analyze the consistency of the traditional mathematical model for the so-called *Debye screening problem* (DSP) originally formulated by Debye and Hückel (*DH model* [1]). In particular, we intend to prove that the 3D Poisson equation involved in the DH model does not admit physically acceptable solutions, i.e., solutions which are provided by ordinary functions and are at least continuous in the domain of existence, i.e., are so-called classical (or strong) solutions. For this purpose a modified model is proposed which takes into account the effect of local plasma sheath (i.e., the local domain near test particles where the plasma must be considered discrete). Basic consequences of the model are discussed, which concern the asymptotic properties of the solutions determined both for weakly and strongly-coupled plasmas.

Despite previous attempts to construct approximate or exact solutions to the DH model [10, 11, 12], the related mathematical model appears incomplete and can be shown to be physically unacceptable, due to the neglect of the local plasma sheath. In fact, it is obvious that sufficiently close to the point-particle the so-called weak-field approximation is violated

making the DH model invalid [8]. In the past [13] it was pointed out that in such a case the test particle does not produce any electric field, but only complete charge neutralization by the plasma, thus producing a Debye length which effectively vanishes. Other objections concerned the asserted indeterminacy of the solution for $x = 0$ due to its divergence in the same point [14]. These issues were later addressed in a more general context [15], including the 2D case where complete neutralization cannot be achieved. To recover the correct physical picture the effect of local plasma sheath must be included. Nevertheless, for suitably dense plasmas or in the case of plasma species characterized by very high electric charges (high- Z), such as dusty plasmas, the weak-field approximation may be locally violated. This circumstance, when the effect of finite local plasma sheath is included, occurs if the normalized electrostatic potential $\hat{\Phi}(\rho)$ results of order unit or larger on the boundary of the plasma sheath (produced by at least one of the s plasma species), namely for $\rho = \rho_{os}$.

It is well-known that, in general, the Debye effect occurs provided suitable physical assumptions are introduced. In particular, the plasma must be assumed appropriately close to kinetic Maxwellian equilibrium, in which each particle species is described by a Maxwellian kinetic distribution function carrying finite fluid fields [defined respectively by the number density, temperature and flow velocity (N, T, \mathbf{V})]. In the absence of test particles these fluid fields must be assumed slowly varying in a suitable sense, or constant, both with respect to position (\mathbf{r}) and time (t). In this regard it is important to remark that the appropriate treatment of the plasma sheath surrounding each test particle is essential also for the validity of the mathematical model for the Debye screening problem, i.e., for the existence of classical solutions of the Debye screening problem, which do not exist when letting $x_o = 0$ [8].

Another significant aspect concerns the issue of the absorption of plasma particles by the test particle, which effectively modifies the local charge density of the background plasma species [16, 17, 18]. Since the particle capture mechanism is a manifestly charge-dependent and velocity-dependent phenomenon (in particular it depends on the angular momentum of the incoming particle), it is obvious that in principle it can produce deviations from local Maxwellian equilibrium [19, 20]. However, this phenomenon is expected to become relevant only if the radii of the test particle and of the surrounding plasma sheath (ρ_p and ρ_o) are comparable, i.e., $\rho_p/\rho_o \sim 1$. Instead, its results are negligible when $\rho_p/\rho_o \ll 1$. Since dusty and colloidal plasmas are characterized by typical grain size ρ_p smaller than $10^{-4} - 10^{-5}$ cm and

radius of plasma sheath ρ_o of the order of $10^{-2} - 10^{-3}$ cm, these effects will be considered negligible.

Goal of this work is the analysis of DSP and the definition of a suitably modified mathematical model to take into account the effect of local plasma sheaths in quasi-neutral plasmas. In particular, in Sec. 2 a modified Debye screening problem (modified DSP) is presented. In Sec. 3 the basic mathematical result is presented which concerns the non-existence of classical solutions of DSP. The proof is reached by noting that DSP can be obtained as limit problem obtained from the modified DSP. Basic feature of the approach is the representation of the Poisson equation in integral form. This permits to analyze the asymptotic properties of the solutions of the modified problem in the limit $x_{os} \rightarrow 0^+$. It is found, that the limit solution of the modified DSP for $x_{os} \rightarrow 0^+$ is a distribution which vanishes identically for all $\rho > 0$ and is discontinuous in $\rho = 0$. Hence, the limit function $\widehat{\Phi}(x)$ is not an strong solution of the DSP equation. This is therefore a characteristic property of the DH model. In particular, as a basic consequence, the effective charge of the DH asymptotic solution c vanishes identically in such a limit and results independent of the charge of the test particle.

II. THE MODIFIED DSP

The traditional formulation of the DSP, based on the Debye-Hückel model [1] regards the test particles as point-like and having a spherically-symmetric charge distribution while ignoring the effect of local plasma sheath. This implies, from the physical standpoint, to neglect the discrete nature of the plasma. Here we shall consider a modified modified Debye-Hückel model, based on the introduction of the notion of *local plasma sheath* [8]. In the sequel we shall consider for simplicity of notation the case of a two species-plasma, formed by electrons and Hydrogen ions, having an unique plasma sheath. Thus, we shall assume that the test particle is represented by a spherically symmetric charge of radius ρ_p . For a particle in which $\rho_p < \rho_o$ the plasma sheath is represented by the spherical shell centered at the position (center) of the test particle for which $\rho_p \leq \rho < \rho_o$, in which the plasma charge density (except for the presence of the test particle) results negligible. In the sequel we can also let in particular $\rho_p = 0$ (point-like test particle) or $\rho_p = \rho_o$ (finite-size test particle). The customary DH model is thus recovered letting $\rho_p = 0$ and taking the

limit $\rho_o \rightarrow 0$ (or in dimensionless variables, requiring $x_p \equiv \rho_p/\lambda_D = 0$ and $x_o \equiv \rho_o/\lambda_D \rightarrow 0$). Denoting $\widehat{\Phi}_{x_o}(x)$ the solution of the Poisson equation, here we intend to determine its asymptotic properties in the limit $x_o \rightarrow 0^+$, while also letting $x_p = 0$ (see Lemma). As a consequence and in agreement with [13, 15], in such a case it follows that the limit function $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) \equiv \widehat{\Phi}(x)$ vanishes identically for $x > 0$, i.e., $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) = 0$. In addition, in the same set we intend to prove the identity

$$\beta - \lim_{x_o \rightarrow 0^+} \int_{x_o}^x dx' x'^2 \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o) = 0, \quad (1)$$

where $\widehat{\Theta}(x - x_o)$ is the (weak) Heaviside function. In detail the relevant equations valid in each subdomain for the normalized electrostatic potential $\widehat{\Phi}_{x_o}(x)$ are as follows. In the internal domain $0 \leq x < x_p$ the electrostatic potential is assumed constant, namely $\widehat{\Phi}_{x_o}(x) = \widehat{\Phi}_{x_o}(x_p)$. In the plasma sheath $x_p \leq x < x_o$, $\widehat{\Phi}_{x_o}(x)$ satisfies the customary Poisson equation in the presence of the charge density produced by a finite-size spherically-symmetric charge

$$\nabla_x^2 \widehat{\Phi}_{x_o} = -\frac{\beta}{x^2} \delta(x - x_p). \quad (2)$$

Finally, in the external domain $x > x_o$ there holds the Poisson equation in the presence of the plasma charge density:

$$x^2 \widehat{\Phi}_{x_o} = \widehat{\Theta}(x - x_o) \sinh \widehat{\Phi}_{x_o}. \quad (3)$$

The boundary conditions, imposed respectively at infinity and at the boundary of the plasma sheath, are specified as follows

$$\lim_{x \rightarrow \infty} \widehat{\Phi}_{x_o}(x) = 0, \quad (4)$$

$$x^2 \frac{d}{dx} \widehat{\Phi}_{x_o}(x) \Big|_{x=x_o} = -\beta. \quad (5)$$

We notice that, if $x_p < x_o$ (for example, $x_p = 0$), $\widehat{\Phi}_{x_o}(x)$ results by assumption at least of class $C^{(1)}(\mathbb{R}_{\{x_p\}})$, where $\mathbb{R}_{\{x_p\}} \equiv]x_p, \infty[$. Here x_o, β are both assumed constant and strictly positive real numbers. The problem defined by (3),(2), together with the boundary conditions (4),(5), will be here denoted as *modified DSP*. From the physical standpoint Eqs.(3),(2) may be viewed as the Poisson equation for a spherical ideally conducting charge, or for a point particle in the presence of a plasma sheath, of radius r_o (i.e., $x_o = r_o/\lambda_D$ in non-dimensional variables) which is in electrostatic equilibrium and is immersed in a spatially uniform quasi-neutral and Maxwellian plasma. As for the previous DSP equation, it follows

that, for solutions satisfying the boundary conditions (4),(5), in the domain $x \in \mathbb{R}_{\{x_o\}}$ Eq.(3) can be cast in the integral form

$$\begin{aligned} \widehat{\Phi}_{x_o}(x) = & \frac{\beta \widehat{\Theta}(x - x_o)}{x} - \\ & - \left[\frac{1}{x} \int_{x_o}^x dx' x'^2 + \int_x^\infty dx' x' \right] \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o). \end{aligned} \quad (6)$$

In particular, thanks to continuity at $x = x_o$ of $\widehat{\Phi}_{x_o}(x)$, one obtains the constraint

$$\widehat{\Phi}_{x_o}(x_o) = \Gamma - \int_{x_o}^\infty dx' x' \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o), \quad (7)$$

with $\Gamma \equiv \frac{\beta}{x_o}$ denoting the Coulomb coupling parameter. It is immediate to establish, the existence and uniqueness of $\widehat{\Phi}_{x_o}(x)$ in the functional class $\widehat{C}^{(\infty)}(\mathbb{R}_{\{x_o\}})$, together with its continuous dependence on initial data, in particular the continuity with respect to the parameter $x_o \in_{\{0\}}$. Moreover, assuming that the weak-fields approximation applies (this condition is manifestly fulfilled identically in the weak-coupling ordering, $0 < \Gamma \sim O(\varepsilon) \ll 1$), and is satisfied at least for $x \gg 1$ suitably large, it is immediate to prove that in this subset an asymptotic solution of the modified DSP is provided by the *external asymptotic solution*

$$\widehat{\Phi}_{x_o}(x) \cong \widehat{\Phi}_{x_o}^{(ext)}(x) \equiv \frac{c}{x} e^{-x+x_o}. \quad (8)$$

Here denoted as of the modified DSP and $c = c(x_o, \Gamma)$ is the *effective dimensionless charge*. Hence, $\widehat{\Phi}_{x_o}^{(ext)}(x)$ reduces formally to the so-called DH potential when $x \gg x_o$. In the weak-coupling ordering it follows $c(x_o, \Gamma) = \frac{\beta}{1+x_o}$, while for strongly-couple plasmas a lower value is expected. Furthermore, it is obvious that the limit function $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x)$ coincides with the solution of DSP, i.e.,

$$\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) = \Phi(x). \quad (9)$$

III. NON-EXISTENCE OF CLASSICAL SOLUTIONS OF DSP

Let us now analyze, which consequences can be obtained for the Debye-Hückel problem, formally obtained by letting $x_o = 0$ in the previous equations [in particular Eq.(6)]. This requires the knowledge of the asymptotic properties of the solution $\widehat{\Phi}_{x_o}(x)$ in the limit $x_o \rightarrow 0^+$. The following Lemma will be invoked [8]:

A. LEMMA - Asymptotic properties of $\widehat{\Phi}_{x_o}(x)$

For any strong solution of the modified DSP, $\widehat{\Phi}_{x_o}(x)$ obtained letting $x_p = 0$, the limit function $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x)$ has the following properties:

1) There results:

$$\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x_o) = +\infty; \quad (10)$$

2) for any $x > 0$ the integral limit (1) is satisfied by $\widehat{\Phi}_{x_o}(x)$. This implies that the limit function $\widehat{\Phi}(x) = \lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x)$ results such for any $x > 0, x \in \{0\}$

$$\widehat{\Phi}(x) = \lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) = 0. \quad (11)$$

3) the following limit is satisfied by the boundary value $\widehat{\Phi}_{x_o}(x_o)$

$$\lim_{x_o \rightarrow 0^+} x_o \widehat{\Phi}_{x_o}(x_o) = 0. \quad (12)$$

4) the limit value of the effective dimensionless charge $c(x_o, \Gamma)$ for $x_o \rightarrow 0^+$, obtained keeping Γ finite, is

$$\lim_{x_o \rightarrow 0^+} c(x_o, \Gamma) = 0. \quad (13)$$

PROOF

In fact, as a consequence of the integral equation (7) and the continuous dependence of $\widehat{\Phi}_{x_o}(x)$ on the initial data, it follows

$$\lim_{x_o \rightarrow 0^+} x_o \widehat{\Phi}_{x_o}(x_o) = \beta - \lim_{x_o \rightarrow 0^+} x_o \int_{x_o}^{\infty} dx' x' \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o), \quad (14)$$

which implies

$$\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x_o) = \infty, \quad (15)$$

$$\lim_{x_o \rightarrow 0^+} \int_{x_o}^{\infty} dx' x' \sinh \widehat{\Phi}_{x_o}(x) \widehat{\Theta}(x' - x_o) = \infty, \quad (16)$$

i.e., the limit function $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x)$ diverges in $x = x_o$. Therefore, due to the continuity of $\widehat{\Phi}_{x_o}(x)$ with respect to $x \in [0, \infty[$ it follows that infinitesimally close to $x, x_o = 0$, and when x, x_o are infinitesimal of the same order, $\widehat{\Phi}_{x_o}(x)$ must diverge logarithmically as

$$\widehat{\Phi}_{x_o}(x) \sim \ln \left\{ \frac{1}{x^3} \right\}. \quad (17)$$

Let us now consider the implications of the integral equation (6) for the limit function $\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x)$ for arbitrary $x \in \{x_o\}$. There follows

$$\begin{aligned} \lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) = & \frac{\beta}{x} - \frac{1}{x} \lim_{x_o \rightarrow 0^+} \int_{x_o}^x dx' x'^2 \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o) - \\ & - \int_x^\infty dx' x' \lim_{x_o \rightarrow 0^+} \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o), \end{aligned} \quad (18)$$

where, due to the asymptotic estimate (17), the second term on the r.h.s. necessarily diverges

$$\lim_{x_o \rightarrow 0^+} \frac{1}{x} \int_{x_o}^x dx' x'^2 \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o) = \infty \quad (19)$$

unless there results for any $x \neq x_o$, $x \in \{x_o\}$

$$\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) = 0. \quad (20)$$

As a consequence of Eq.(20), from the integral equation (6) it follows necessarily that for all $x > 0$:

$$\lim_{x_o \rightarrow 0^+} \frac{\beta \widehat{\Theta}(x - x_o)}{x} = \lim_{x_o \rightarrow 0^+} \frac{1}{x} \int_{x_o}^x dx' x'^2 \sinh \widehat{\Phi}_{x_o}(x') \widehat{\Theta}(x' - x_o), \quad (21)$$

which proves the limit (1). As a consequence it must result necessarily that the limit $\lim_{x_o \rightarrow 0^+} \sinh \widehat{\Phi}_{x_o}(x)$ is a Dirac delta. The limit (12) follows immediately from the boundary condition (7), while Eq.(20) implies manifestly the limit (13).

As an immediate consequence of the Lemma it follows that the DSP equation obtained letting $x_o = 0$ in Eq.(6) does not admit classical solutions.

B. THEOREM - Non-existence of classical solutions of DSP

In the functional class $\widehat{C}^{(2)}(\{0\})$ the DSP problem defined by Eqs.((6) and the boundary conditions indicated above [Eqs.(BC-1 b)(BC-2 b)] does not admit strong solutions.

PROOF

In fact, first, we notice that the limit function

$$\lim_{x_o \rightarrow 0^+} \widehat{\Phi}_{x_o}(x) \equiv \widehat{\Phi}(x), \quad (22)$$

is manifestly a solution of the DSP equation which satisfies the required boundary conditions (4,refBC-2 b). On the other hand, due to the Lemma, this solution is discontinuous in $x = x_o$ and results a distribution. Hence it is not a strong (classical) solution of the modified DSP problem.

The basic implication of the Lemma and the theorem is that the DSP equation, provided by the DH model, must be regarded as physically unacceptable, since it does not admit strong solutions. In this regard it should be noted that, as a basic principle, physically acceptable solutions of ordinary (or partial) differential equations characterizing the classical theory of fields must be suitably smooth strong solutions. The modified Debye screening problem here defined, instead, exhibits smooth strong solutions and therefore appears, from this viewpoint, consistent.

IV. CONCLUSIONS

The essential implication of the present result is that the customary mathematical model used for the investigation of the Debye screening problem is incorrect. The correct mathematical model requires, in fact, the treatment of the local plasma sheath, for which a simple model is provided by Eq.(6). Important physical consequences follow. These concern the correct estimate of the Debye screening effect, which occurs close to the local plasma sheath and, particularly, for highly-charged test particles immersed in strongly-coupled plasmas. In fact, the modified DSP can be used to obtain asymptotic estimates for the effective dimensionless charge $c(x_o, \Gamma)$ carried by the DH potential in strongly-coupled plasmas [9]. The resulting charge screening effect appears produced by non-linear effects in the Poisson equation. As a consequence, outside the Debye sphere (i.e., for $x > 1$) the DH potential generated by highly charged test particles in strongly-coupled plasmas results strongly reduced with respect to the theoretical value observed in the corresponding weakly-coupled systems.

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Figure captions

Figure 1 - Comparison between $f_c(x)$, $c(x_o, \Gamma)$ and $c^{(a)\prime}(x_o, \Gamma)$. The data are normalized with respect to β , the normalized charge of the isolated test particle. The figure concerns the case with $\beta = 1$ and $x_o = 0.05$, yielding $\Gamma = 20$. The horizontal straight line represents the asymptotic estimate $c^{(a)\prime}(x_o, \Gamma)$, while the curve below it is the graph of $f_c(x)$. It follows $f_c(x_o) \cong 0.564$, while the asymptotic value $c(x_o, \Gamma) \cong 0.493$ is reached approximately at $x \approx 0.3$, and the upper bound for the normalized effective charge is $c^{(a)\prime}(x_o, \Gamma) \cong 0.589$.

Figure 2 - Comparison between $f_c(x)$, $c(x_o, \Gamma)$ and $c^{(a)\prime}(x_o, \Gamma)$ for $\beta = 5$ and $x_o = 0.2$ (with $\Gamma = 25$). In this case $f_c(x_o) \cong 0.369$, while the asymptotic value $c(x_o, \Gamma) \cong 0.28$ is reached approximately at $x \approx 0.4$, and $c^{(a)\prime}(x_o, \Gamma) \cong 0.38$.

Figure 3 - Comparison between $f_c(x)$, $c(x_o, \Gamma)$ and $c^{(a)\prime}(x_o, \Gamma)$ for $\beta = 10$ and $x_o = 0.3$ (with $\Gamma \cong 33$). In this case it is found $f_c(x_o) \cong 0.273$, while the asymptotic value $c(x_o, \Gamma) \cong 0.188$ is reached approximately at $x \approx 0.5$, and $c^{(a)\prime}(x_o, \Gamma) \cong 0.303$.

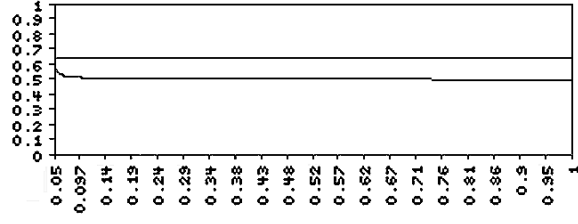


FIG. 1:

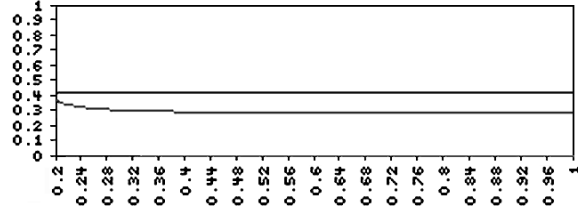


FIG. 2:

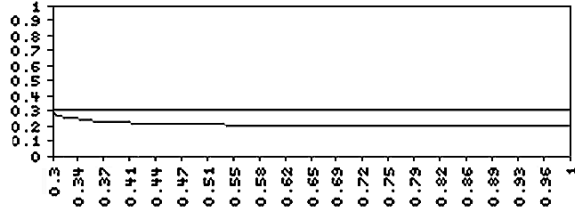


FIG. 3: